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Nonlinear Oscillations of Fixed Period

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I. INTRODUCTION

In [1] Urabe considers the nonlinear oscillator

$$\frac{d^2x}{dt^2} + g(x) = 0 \tag{1.1}$$

and finds a necessary and sufficient condition for all the solutions of (1.1) near $x = x' = 0$ to oscillate around $x = x' = 0$ with the same period $\omega > 0$. He shows by an example that $g(x) = kx$ $(k > 0)$ is not such a condition.

Here we consider the same problem and obtain the following result, which is then compared with Urabe's.

THEOREM. *Let $g(x) \in C$ for x sufficiently small and let $xg(x) > 0$ for $x \neq 0$. Define $G(x)$, $X(x)$, and $h(X)$ by*

$$G(x) = \int_0^x g(\xi)\, d\xi \tag{1.2}$$

$$\tfrac{1}{2} X^2(x) = G(x) \textit{ and } \frac{1}{x} X(x) > 0 \textit{ for } x \neq 0 \tag{1.3}$$

$$h(X) = g(x(X)), \tag{1.4}$$

where $x(X)$ is the inverse function of $X(x)$.

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Then every solution of (1.1) *near* $x = x' = 0$ *oscillates around* $x = x' = 0$ *with the same period* $\omega > 0$ *if and only if*

$$h(X) = \frac{2\pi}{\omega} \frac{X}{1 + S(X)} \qquad (X \neq 0) \tag{1.5}$$

for X *sufficiently small, where* $S(X)$ *is an odd function, defined and continuous for sufficiently small* $X \neq 0$, *and* $S(X) \in L_1(0, \epsilon)$ *for some* $\epsilon > 0$.

The proof depends upon the following lemma which is proved at the end of the paper.

LEMMA. *If* $T(X) \in C(0, R_0]$, $T(X) \in L_1(0, R_0)$, *and*

$$\int_0^{\pi/2} T(R \cos \varphi)\, d\varphi = 0 \qquad (0 < R \leqslant R_0), \tag{1.6}$$

then

$$T(X) = 0 \qquad (0 < X \leqslant R_0). \tag{1.7}$$

The above theorem generalizes that of [1] in two ways:

(i) In [1], (1.5) is replaced by

$$h(X) = \frac{2\pi}{\omega} \frac{X}{1 + S(X) + T(X)}, \tag{1.8}$$

where, among other things, $S(X)$ is an odd continuous function (for all sufficiently small X), and where $T(X)$ is an even continuous function satisfying (1.6) (for all sufficiently small X). By the preceding lemma, such a $T(X) \equiv 0$ so that (1.8) reduces to (1.5). (In [1] it was shown that $T(X) \equiv 0$ if $g(x)$ is analytic.)

(ii) In [1], (1.8) is derived under the assumption that $g(x) \in C'$, whereas here (1.5) is derived under the weaker hypothesis $g(x) \in C$. This necessitates, apart from the use of the Lemma, a different proof—which is simpler than that of [1]. The two approaches have the definitions (1.3) and (1.4) in common, and these are at the heart of the matter.

The following observation (which, essentially, appears in [1]) is helpful in comparing our Theorem with Urabe's. If $g(x) \in C$ for x sufficiently small, then $xg(x) > 0$ $(x \neq 0)$ is a necessary and sufficient condition for all solutions of (1.1) sufficiently close to $x = x' = 0$ to oscillate around $x = x' = 0$. This remark may be used to inessentially modify the statement of the Theorem in such a way as to avoid the a priori hypothesis $xg(x) > 0$ $(x \neq 0)$.

As a consequence of the Theorem it is shown that

COROLLARY. *Let $g(x) \in C$ for x sufficiently small, $xg(x) > 0$ for $x \neq 0$, and $g(x)$ be an odd function. Then every solution of (1.1) near $x = x' = 0$ oscillates around $x = x' = 0$ with the same period $\omega > 0$ if and only if*

$$g(x) = \left(\frac{2\pi}{\omega}\right)^2 x \tag{1.9}$$

for x sufficiently small.

In [1] there is a similar result but with the additional hypothesis that $g(x)$ be analytic. The proofs are essentially the same, as the Corollary is an easy consequence of (1.5) and $g(x)$ odd. (As already noted, (1.5) was obtained in [1] for analytic $g(x)$.)

II. Proof of the Theorem

It is clear from the hypothesis that $h(X) \in C$ for X sufficiently small and that $X/h(X) > 0$ for $X \neq 0$. In the usual way (see, e.g., [2]) one has

$$\tau(E) = \sqrt{2} \int_{-B}^{A} (E - G(x))^{-1/2}\, dx \tag{2.1}$$

for A, $B > 0$ sufficiently small, where $E = G(A) = G(-B)$, and where $\tau(E)$ is the period of the solution of (1.1) of energy E. Making the change of variables (1.3) in (2.1) yields

$$\tau(E) = 2 \int_{-(2E)^{1/2}}^{(2E)^{1/2}} (2E - X^2)^{-1/2} \frac{X}{h(X)}\, dX \qquad (0 < E \leqslant E_0) \tag{2.2}$$

for some $E_0 > 0$. It is clear from (2.2) and the preceding that

$$\frac{X}{h(X)} \in C(0 < |X| \leqslant (2E_0)^{1/2}) \cap L_1(-(2E_0)^{1/2}, (2E_0)^{1/2}).$$

(a) We first prove the necessity of (1.5). From (2.2) and the hypothesis one has

$$\omega = 2 \int_{-(2E)^{1/2}}^{(2E)^{1/2}} (2E - X^2)^{-1/2} \frac{X}{h(X)}\, dX \qquad (0 < E \leqslant E_0). \tag{2.3}$$

Define

$$V(X) = \frac{2\pi}{\omega} \frac{X}{h(X)} - 1 \tag{2.4}$$

$$S(X) = \tfrac{1}{2}(V(X) - V(-X)),\ T(X) = \tfrac{1}{2}(V(X) + V(-X)) \tag{2.5}$$

for $0 < |X| \leqslant (2E_0)^{1/2}$. Then $S(X)$ is odd, $T(X)$ is even, and

$$V(X),\, S(X),\, T(X) \in C(0 < |X| \leqslant (2E_0)^{1/2}) \cap L_1(-(2E_0)^{1/2}, (2E_0)^{1/2}).$$

From (2.3), (2.4), and (2.5) one obtains

$$\int_0^{(2E)^{1/2}} (2E - X^2)^{-1/2}\, T(X)\, dX = 0 \qquad (0 < E \leqslant E_0).$$

The change of variables $X = (2E)^{1/2} \cos \varphi$ yields

$$\int_0^{\pi/2} T((2E)^{1/2} \cos \varphi)\, d\varphi = 0 \qquad (0 < E \leqslant E_0).$$

The Lemma now implies $T(X) = 0$ for $0 < |X| \leqslant (2E_0)^{1/2}$, which together with (2.4), (2.5) yields (1.5).

(b) To prove the sufficiency of (1.5), one has only to substitute (1.5) into (2.2) and obtain $\tau(E) = \omega$ $(0 < E \leqslant E_0)$.

III. Proof of the Corollary

The sufficiency of (1.9) is well known; we prove the necessity. As $g(x)$ is odd, $G(x)$ is even. Together with (1.3), this shows that $X(x)$, and hence $x(X)$, is odd. From (1.4), it now follows that $h(X)$ is odd. Consequently, (1.5) implies that $S(X) = 0$ $(X \neq 0)$. As $h(X)$ is continuous, one has

$$g(x(X)) = h(X) = \frac{2\pi}{\omega} X \tag{3.1}$$

for X sufficiently small. From (1.3) it follows that $X(x)X'(x) = g(x)$ $(x \neq 0)$. This together with (3.1) and the continuity of $X(x)$ implies that $X(x) = (2\pi/\omega)x$ for x sufficiently small. The latter together with (3.1) implies (1.9) and completes the proof.

IV. Proof of the Lemma

Setting $X = R \cos \varphi$ in (1.6) yields

$$\int_0^R (R^2 - X^2)^{-1/2}\, T(X)\, dX = 0 \qquad (0 < R \leqslant R_0). \tag{4.1}$$

We now show by induction that

$$\int_0^R (R^2 - X^2)^{(2n+1)/2} T(X)\, dX = 0 \qquad (0 < R \leqslant R_0;\, n = -1, 0, 1, \cdots). \tag{4.2}$$

From (4.1) one has (4.2) for $n = -1$. Suppose (4.2) is true for $n = k \geqslant -1$. Then

$$0 = \int_0^R S \left\{ \int_0^S (S^2 - X^2)^{(2k+1)/2} T(X)\, dX \right\} dS$$

$$= \frac{1}{2k+3} \int_0^R (R^2 - X^2)^{(2k+3)/2} T(X)\, dX \qquad (0 < R \leqslant R_0),$$

which establishes (4.2) for $n = k + 1$ and completes the induction.

Setting $X = R \cos \varphi$ in (4.2) yields

$$\int_0^{\pi/2} \sin^{2n} \varphi\, T(R \cos \varphi)\, d\varphi = 0 \qquad (0 < R \leqslant R_0;\, n = 0, 1, \cdots). \tag{4.3}$$

An elementary induction shows that $\cos 2n\varphi$ is a polynomial in $\sin^2 \varphi$ for any integer n. Consequently, (4.3) implies

$$\int_0^{\pi/2} \cos 2n\varphi\, T(R \cos \varphi)\, d\varphi = 0 \qquad (0 < R \leqslant R_0;\, n = 0, 1, \cdots),$$

so that

$$\int_0^{\pi} \cos n\varphi\, T\left(R \cos \frac{\varphi}{2}\right) d\varphi = 0 \qquad (0 < R \leqslant R_0;\, n = 0, 1, \cdots). \tag{4.4}$$

It now follows from (4.4), the hypothesis on $T(X)$, and well-known facts about Fourier series that, for each fixed R in $0 < R \leqslant R_0$, the function $T(R \cos \varphi/2) \equiv 0$ for $0 \leqslant \varphi < \pi$. This implies (1.7).

Note Added in Proof: Recently Urabe (*Arch. Rat. Mech. Anal.* **11**, 27-33 (1962) and *J. Sci. Hiroshima Univ., Ser. A*-1, **26**, 93-109, 111-122 (1962)) has greatly extended the problem and results of [1]. His methods, however, seem to require a little more smoothness on $g(x)$ than mere continuity—in order to obtain (1.5) and the Corollary above.

References

1. Urabe, M., Potential forces which yield periodic motions of a fixed period. *J. Math. Mech.* **10**, 569-578 (1961).
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